# A Quantum Mechanical Derivation of Gamow's Relation
# For the Time and Temperature of the Expanding Universe

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## Abstract

The quantum mechanical approach developed by us recently for the evolution of the universe is used to derive an alternative derivation connecting the temperature of the cosmic background radiation and the age of the universe which is found to be similar to the one obtained by Gamow long back. By assuming the age of the universe to be $\approx 20$ billion years, we reproduce a value of $\approx 2.91$ K for the cosmic back-ground radiation, agreeing well with the recently measured experimental value of 2.728 K. Besides, this theory enables us to calculate the photon density and entropy associated with the background radiation and the ratio of the number of photons to the number of nucleons, which quantitatively agree with the results obtained by others.

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It has been, by now, accepted that the most important theory for the origin of the universe is the Big Bang Theory [1], according to which the present universe is considered to have started with a huge explosion from a superhot and a superdense stage. Theoretically one may visualize its starting from a mathematical singularity with infinite density. This also comes from the solutions of the type I and type II form of Einstein's field equations [2]. What follows from all these solutions is that the universe has originated from a point where the scale factor $R$ (to be identified as the radius of the universe) is zero at time $t = 0$, and its derivative with time is taken to be infinite at this time. That is, it is thought that the initial explosion had happened with infinite velocity, although, it is impossible for us to picture the initial moment of the creation of the universe.

An indication in support of the Big Bang theory is the expansion of the universe, which has been established by means of the Hubble's law [3] as

$$v = H_0 d \ , \tag{1}$$

where $v$ is the radial recession velocity of the galaxy and $d$ is the distance of the galaxy from us and $H_0$ is known as the Hubble constant. The quantity $(1/H_0)$, which is known as the Hubble time, is a measure for the maximum age of the universe. Since, it is very difficult to correctly determine the distance $d$ to the galaxies, there is a great uncertainty in the estimated value of $H_0$. The correct value for the age of the universe seems to lie in between 10 to 20 billion years. The most important evidence for the Big Bang theory is the microwave background radiation, which was discovered by Penzias and Wilson [4] with an effective temperature of $\approx 3.5K$. However most recent measurement on the cosmic background radiation using Far Infrared Absolute Spectrometer (FIRAS) [5] has yielded a value of 2.728 K. The characteristic of this radiation is that it is almost absolutely isotropic, that is, it comes to us from all directions with the same intensity.

This means that the radiation is not due to stars or galaxies which are the measure of the inhomogeneities of the universe. The only plausible explanation for the origin of the cosmic background radiation is that the universe has, perhaps, passed through a state of very high density and high temperature in its early state, and the present temperature of $\approx 2.728K$ [5] is nothing but the remnant of the intense heat of the Big Bang, which has been redshifted into the microwave region.

Since, within a time period very close to the Big Bang explosion $(t = 0)$, the universe was lying in its 'radiation dominated era', there was no possibility for the formation of elementary particles with finite mass at that stage. The actual creation of material particles must have taken place a few seconds after the Big Bang. In the mid 1970s, Gamow [6] suggested that the high density and high temperature required for the synthesis of elements, existed in the few moments after the Big Bang. In his simplified picture, Gamow assumed the universe to be intially made of neutrons and photons. As one knows, the neutrons are charge free particles found in the nuclei of atoms, while photons are the quanta of the electromagnetic field that constitute light. Gamow arrived at a relation, relating the temperature $T$ of the universe with time $t$ after its Big Bang, which is given as

$$T = \left( \frac{3c^2}{32\pi Ga} \right)^{1/4} t^{-1/2} K \ , \tag{2a}$$

where $a$ is the radiation constant and all other constants in the above equation have their usual meaning. A numerical estimate of the factor within the bracket in the above equation gives

$$T = 1.5 \times 10^{10} \ t^{-1/2} K \ . \tag{2b}$$

Later on, the above relation was modified by Hayashi [7] by taking into account the effects of the thermal equilibrium among the particles, like neutrons, protons, electron-positron pairs and neutrino-antineutrino pairs, present in the universe at the very temperatures

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that existed right at the beginning of the universe. Hayashi obtained a relation, which is given as

$$T = 10^{10} t^{-1/2} K .\qquad(3)$$

As one can see from Eq. (3), it differs from Gamow's derivation [Eq. (2b)] with respect to the extra factor of (3/2). Based on the relation (2b), Gamow made the prediction that a very faint back-ground of radiation, known as the relic of the Big Bang, should exist at the present epoch of the universe. This was subsequently verified by Penzias and Wilson [4], who reported an isotropic radiation background with a temperature of $\approx 3.5K$, in the microwave region. If one takes Eq. (3) to be correct, then, to reproduce a temperature of $\approx 2.728K$ the present age of the universe would be $\approx 425$ billion years instead of 20 billion years, where the latter one has been known to be very close to the accepted value. Gamow's relation would give a value of $\approx 956$ billion years for the age, which is much more absurd.

Recently, we have developed a quantum mechanical theory [8] for a system of self-gravitating particles like the stars that have exhusted all the nuclear power at their respective cores. In these systems, the particles interact with each other gravitationally. Using a singular form of single particle density to account for the distribution of particles within the system, we have been able to obtain a compact expression for the radius of a neutron star. Comparing this with the Schwarzschild radius, we arrive at a critical value [8] for the mass of the neutron star beyond which it should go over to the stage of a blackhole. Our value for the critical mass, seems to agree with those of other theoretical calculations. Applying such a theory to a white dwarf, we have succeeded to reproduce the so called Chandrasekhar limit [9] for its critical mass. In a subsequent work, we have used the above theory [10] to make a study about on the evolution of the present universe by visualizing it as a system constituted of a large number of self-gravitating

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ficticious particles, fermionic in nature, interacting with each other through gravitational potentials. As far as the neutron stars are concerned, they are obviously constituted of fermions. For the universe, it is being said that the major constituent of the total mass of the present universe is made of the Dark Matter (DM). Since neutrinos are considered to be the most probable candidates for the particles of the DM, we are justified to say that the universe is constituted of particles that are fermions. Proceeding in a manner similar to the one used by us [8] for the study of stars, we arrive at an expression [10] for the radius of the universe which, after invoking the Mach principle [11], assumes a form involving only the fundamental constants like $G$, $\hbar$ , $c$ and the mass $m$ of the constituent particles. Following this expression, we have made an estimate of the total mass of the universe, which is found to be agreeing with the results of other theoretical calculations [12]. Our calculated value for $(\dot{G}/G)$ is also in good agreement with those of many earlier workers. There are many other interesting results that follow from this theory, which demand to have a deeper study on the subject. In this present paper, we want to apply the very theory [10] to make an estimate of the temperature of the cosmic background radiation, whose most recent value has been reported to be $\approx 2.728K$ [5]. Using our expression, we also try to discuss about the production of the various elementary particles that took place in the early universe. All these have been dealt with, in the next section.

The Hamiltonian used by us recently [8,10] for the study of a system of self-gravitating particles is written as

$$H = -\sum_{i=1}^{N}(\frac{\hbar^2}{2m})\nabla_i^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,i\neq j}^{N} v(|\ \vec{X}_i - \vec{X}_j\ |)\ , \tag{4}$$

where $v(|\ \vec{X}_i - \vec{X}_j\ |) = -g^2/\ |\ \vec{X}_i - \vec{X}_j\ |$, having $g^2 = Gm^2$, $G$ being the universal gravitational constant and $m$ the mass of the constituent particles whose number is $N$. Since the measured value for the temperature of the cosmic microwave background radiation is

$\approx 2.728K$, it lies in the neighbourhood of almost zero temperature. We,therefore, use the zero temperature formalism for the study of the present problem. Under the situation $N$ is extremely large, the total kinetic energy of the system is obtained as

$$< KE >= \left(\frac{3\hbar^2}{10m}\right)(3\pi^2)^{2/3} \int d\vec{X}[\rho(\vec{X})]^{5/3} \, , \qquad (5a)$$

where $\rho(\vec{X})$ denotes the single particle density to account for the distribution of particles (fermions) within the system, which is considered to be a finite one. Eq. (5a) has been written in the Thomas-Fermi approximation. The total potential energy of the system, in the Hartree-approximation, is now given as

$$< PE >= -(\frac{g^2}{2}) \int d\vec{X} d\vec{X}' \frac{1}{\mid \vec{X} - \vec{X}' \mid} \rho(\vec{X})\rho(\vec{X}') \, . \qquad (5b)$$

Inorder to evaluate the integral in Eq. (5a) and Eq. (5b), we had chosen a trial single-particle density $\rho(\vec{X})$ [8,10] which was of the form :

$$\rho(\vec{X}) = \frac{Ae^{-x}}{x^3} \, , \qquad (6)$$

where $x = (r/\lambda)^{1/2}$, $\lambda$ being the variational parameter. As one can see from Eq. (6), $\rho(\vec{X})$ is singular at the origin. Its interpretation has already been given in our earlier papers [8,10]. After evaluating the integrals in Eq. (5) to find the total energy $E(\lambda)$ of the system, we minimize it with respect to $\lambda$ and thereby we obtain the value of the energy of the system corresponding to its lowest energy state (ground state). Following the expression for $< KE >$ evaluated at $\lambda = \lambda_{min} = \lambda_0$, we write down the value of the equivalent temperature $T$ of the system, using the relation

$$\begin{aligned} T =& (\frac{2}{3})(\frac{1}{k_\beta})[\frac{< KE >}{N}] \\ =& (\frac{2}{3})(\frac{1}{k_\beta})(0.015442)N^{4/3}(\frac{mg^4}{\hbar^2}) \end{aligned} \qquad (7)$$

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The expression for the radius $R_0$ of the universe, as found by us earlier [10], is given as

$$R_0 = 4.047528(\frac{\hbar^2}{mg^2})/N^{1/3} \; . \tag{8}$$

After invoking Mach's principle [11], which is expressed through the relation $(\frac{GM}{R_0 c^2}) \approx 1$, and using the fact that the total mass of the universe $M = Nm$, we are able to obtain the total number of particles $N$ constituting the universe, as

$$N = 2.8535954(\frac{\hbar c}{Gm^2})^{3/2} \; . \tag{9}$$

Now, substituting Eq. (9) in Eq. (8), we arrive at the expression for $R_0$, as

$$R_0 = 2.8535954(\frac{\hbar}{mc})(\frac{\hbar c}{Gm^2})^{1/2} \; . \tag{10}$$

As one can see from above, $R_0$ is of a form which involves only the fundamental constants like $\hbar, c, G$ and $m$. Now, eliminating $N$ from Eq. (7), by virtue of Eq. (9),we have

$$T = \frac{2}{3}(0.0625019)(\frac{mc^2}{k_\beta}) \; . \tag{11}$$

Let us now assume that the radius $R_0$ of the universe is approximately given by the relation

$$R_0 \simeq ct \; , \tag{12}$$

where $t$ denotes the age of the universe at any instant of time. The Hubble's law as indicated in Eq. (1), also implies that the universe is expanding uniformly. Although, it is so for the universe, all the galaxies are not uniformly expanding. Considering a photon of light with wave length $\lambda$ travelling a distance of separation 'd' between two galaxies at rest with respect to each other, one has $d = ct$, where 't' is the time it takes for light to travell the space between the galaxies. Because of the expansion of the universe, the galaxies move away from each other at a velocity $v$ known as the radial velocity.

During this time 't', the galaxies are separated by a distance $\Delta d$ given by $\Delta d = vt$. Thus, one finds that $\frac{\Delta d}{d} = (\frac{v}{c})$. From this, it follows that the greater is the relative velocities of the galaxies, the greater is the separation attained in the time interval $t$. The importance of Hubble law, as stated through Eq. (1), is that the galaxies were closer in the past than they are now. As we have stated earlier, the Hubble time $(\frac{1}{H_0})$ represents the maximum age of the universe, because the galaxies themselves slow down the expansion of the universe. Even though the galaxies are farther apart, they still exert a gravitational force on each other. Their mutual gravity continuously acts to pull other galaxies together. This means that the universe was expanding faster in the past than it is now. As indicated in Eq. (12), the velocity of expansion of the universe is being approximated to be equal to the velocity of light 'c'.

Following Eq. (10) and Eq. (12), we write $m$ as

$$m = (\frac{\hbar^3}{Gc^3})^{1/4}(2.8535954)^{1/2}\frac{1}{\sqrt{t}} \ . \tag{13}$$

A substitution of $m$, from the above equation, in Eq. (11), enables us to write

$$T = 0.070388(\frac{1}{k_\beta})(\frac{c^5\hbar^3}{G})^{1/4}t^{-1/2} \tag{14a}$$

$$= 0.070388[(\frac{c^3}{G})\frac{\pi^2}{60\sigma}]^{1/4}t^{-1/2} \ , \tag{14b}$$

where $\sigma = (\frac{\pi^2 k_\beta^4}{60\hbar^3 c^2})$, is the Stefan-Boltzman constant [13]. Substituting the numerical value of $\sigma$, which is equal to $5.669 \times 10^{-5} \ erg/cm^2.deg^4.sec$, and the present value for the universal gravitational constant $G$ [$G = 6.67 \times 10^{-8} dyn.cm^2.gm^{-2}$], in Eq. (14b),we obtain

$$T = (0.23172 \times 10^{10})t^{-1/2}K \ . \tag{15}$$

As one can very well see, Eq. (15) is of the same form as obtained by Gamow [Eq. (2b)] leaving aside the multiplying constant 0.23172. If we accept the age of the universe to

be close to $20 \times 10^9 yr$, which we have used here, with the help of Eq. (15), we arrive at a value for the cosmic background temperature equal to $\approx 2.91K$. This is very close to the measured value of 2.728 K as reported from the most recent Cosmic Background Explorer (COBE) satellite measurements [5]. However, to reproduce the exact value of 2.728 K for the cosmic background temperature from our expression, Eq. (15), we would require an age of $22.832 \times 10^9 yr$ for the universe.

By virtue of the expression given in Eq. (14b), we find

$$\sigma T^4 \simeq 2.4547 \times 10^{-5} \left(\frac{\pi^2 c^3}{60G}\right) \frac{1}{t^2} \ . \tag{16}$$

The very form of the above equation suggests that the factor in its right hand side (rhs) can be identified as the energy density of the electromagnetic radiation at a time $t$. The radiation of this form is belived to follow the black- body law. The very agreement of our calculated result with the most accurate value for the temperature of the background radiation shows that age of the universe is very close to $\approx 20 \times 10^9 yr$. This also creates a kind of confidence in us regarding the correctness of our theory compared to others, inspite of its basic difference from the conventional approaches, relating to the evolution of the universe.

Using Eq. (15), we have made an estimation of the temperature of the universe at various stages of its evolution in time. Comparing the energy, associated with the temperature $T$, with $mc^2$, we calculate the masses of the elementary particles formed at various times. This is shown in table-I of this paper. From the table, we notice that when the age of the universe was less than 5 sec, the formation of electron and positron was possible, while when the age of the universe was less than $1.2 \times 10^{-4}$ sec, the formation of muons and their antiparticles must have taken place. For the formation of mesons and their antiparticles, which needs a temperature of $\approx 1.6 \times 10^{12} K$, the corresponding age of the universe would be less than $\approx 7 \times 10^{-5} sec$ .As far as the nucleon (neutron

and proton) and their antiparticles were concerned, they must have been formed before an age of $1.5 \times 10^{-6}$ sec. Thus, the period between $t = 7 \times 10^{-5}$ sec and 5 sec may be called as the lepton era, while the period before $7 \times 10^{-5}$ sec is called the hadron era. The very early era which is known as the planck era corresponds to the period $t < 10^{-43}$ sec, ($temperature < 10^{32}K$). During this period, gravity is considered to be playing a major role and it is to be, possibly, quantized at that stage.

Having evaluated the expression in the rhs of Eq. (16), the energy of the electromagnetic radiation radiated per unit area per unit time is given as

$$u = 1.6345 \times 10^{33}(\frac{1}{t^2}) \, , \tag{17}$$

where $t$ is the age of the universe in sec at any instant of time. The entropy $S$ associated with the microwave back-ground radiation is obtained as [14]

$$S = \frac{16Vu}{3cT} = 2.9058(\frac{V}{T}) \times 10^{23}(\frac{1}{t^2}) \, . \tag{18}$$

Assuming the present universe to be spherical, its volume $V$ is given as $V = (\frac{4\pi}{3})R_0^3$, where $R_0$ denotes its radius. Taking $R_0 \simeq 2.16 \times 10^{28}$ cm, which corresponds to the age $t = 22.832 \times 10^9 yr$, since ($R_0 \approx ct$), the photonic entropy of the present universe is calculated to give

$$S = 2.369 \times 10^{73}(\frac{1}{T})erg/deg \, , \tag{19a}$$

For $T = 2.728K$, it becomes,

$$S = 0.86 \times 10^{73} \approx 10^{73}erg/deg \, . \tag{19b}$$

The equilibrium number of photons [14] associated with the microwave background radiation is given as

$$\overline{N}_\gamma = \frac{V2\zeta(3)}{\pi^2\hbar^3c^3}k_\beta^3T_0^3 \simeq (410.0)V \, . \tag{20}$$

Following this, the photon density is found to be $(\frac{\overline{N}_\gamma}{V}) \simeq 410$, which is in very good agreement with the estimated value of 400 found [15] by doing a calculation of the total energy density carried by the cosmic microwave background radiation. Using Eq. (20), we have calculated the total number of photons in the present universe, which becomes

$$\overline{N}_\gamma = 1.74 \times 10^{88} \ . \tag{21}$$

Considering the fact that the number of nucleons, $N_n$, in the present universe is $\approx$ $6.30 \times 10^{78}$, [11], we obtain

$$(\frac{\overline{N}_\gamma}{N_n}) \simeq 0.28 \times 10^{10} \ . \tag{22}$$

This agrees with the value $(0.14 \sim 0.33) \times 10^{10}$ as speculated by several earlier workers [16] following calculations on baryogenesis.

To conclude, we find that the theory developed by us recently [10] for the evolution of the universe proves to have its further success in reproducing the temperature of the cosmic background radiation correctly. Besides, it also succeeds to reproduce the photon density associated with the background radiation, and the value of the ratio $(\overline{N}_\gamma/N_n)$, which nicely match with the results predicted by others.

**TABLE - I**

| Age of the universe (t) in sec. | Temperature (T) in K as calculated from Eq. (15) | Temperature (T)in K for the formation of elementary particles |
| --- | --- | --- |
| 5 | $\approx 1 \times 10^9$ | $\approx 6 \times 10^9 (e^+, e^-)$ |
| $1.2 \times 10^{-4}$ | $\approx 2.1 \times 10^{11}$ | $\approx 1.2 \times 10^{12}(\mu^+, \mu^-)$ and their antiparticles |
| $7 \times 10^{-5}$ | $\approx 2.8 \times 10^{11}$ | $\approx 1.6 \times 10^{12}(\pi^0, \pi^+, \pi^-)$ and their antiparticles |
| $1.5 \times 10^{-6}$ | $\approx 1.9 \times 10^{12}$ | $\approx 10^{13}$ (protons, neutron and their antiparticles) |
| $10^{-43}$ | $\approx 7.3 \times 10^{30}$ | $\approx 10^{32}$ (planck mass) |

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